

# ALTERNATING LINKS AND LEFT-ORDERABILITY

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ABSTRACT. Let  $L \subset S^3$  denote an alternating link and  $\Sigma(L)$  its branched double-cover. We give a short proof of the fact that the fundamental group of  $\Sigma(L)$  admits a left-ordering iff  $L$  is an unlink. This result is originally due to Boyer-Gordon-Watson.

## 1. A GROUP PRESENTATION.

Consider a link  $L \subset S^3$  presented by a connected planar diagram. Color its regions black and white in checkerboard fashion, and assign each crossing a sign as displayed in Figure 1. From this coloring we obtain the *white graph*  $W = (V, E)$ . This is the planar graph with one vertex for each white region, one signed edge for each crossing where two white regions touch, and one arbitrary distinguished vertex  $r$  (the *root*).

We form a group  $\Gamma$  as follows. It has one generator  $x_v$  and one relation  $r_v$  for each  $v \in V$ , as well as one additional relation  $x_r$  (no confusion about the  $r$ 's!). To describe the relation  $r_v$ , consider a small loop  $\gamma_v$  centered at  $x_v$  and oriented counter-clockwise. Starting at an arbitrary point along  $\gamma_v$ , the loop meets edges  $(v, w_1), \dots, (v, w_k)$  with respective signs  $\epsilon_1, \dots, \epsilon_k$  in order; then  $r_v = \prod_{i=1}^k (x_{w_i}^{-1} x_v)^{\epsilon_i}$ .

**Proposition 1.1.** *The fundamental group of  $\Sigma(L)$  is isomorphic to  $\Gamma$ .* □

Proposition 1.1 is established in [5, §3.1], in which the presentation for  $\Gamma$  derives from a specific Heegaard diagram of  $\Sigma(L)$  (see the beginning of [5, §3.2], as well). Dylan Thurston points out that the standard derivation of the Wirtinger presentation of a knot group suggests an alternate route to this fact.

## 2. NON-LEFT-ORDERABILITY.

In this section we use Proposition 1.1 to establish the main result.

**Theorem 2.1** (Boyer-Gordon-Watson [1]). *If  $L$  is an alternating link, then  $\pi_1(\Sigma(L))$  admits a left-ordering iff  $L$  is an unlink.*

*Proof.* Observe that if  $L = L_1 \cup L_2$  is a split link, then  $\Sigma(L) \cong \Sigma(L_1) \# \Sigma(L_2)$  and  $\pi_1(\Sigma(L))$  decomposes as the free product  $\pi_1(\Sigma(L_1)) * \pi_1(\Sigma(L_2))$ . Furthermore, a free product admits a left-ordering iff each of its factors do [9]. Therefore, it suffices to restrict attention to the case

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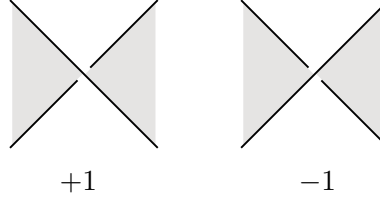


FIGURE 1. Crossings and signs.

of a non-split alternating link  $L$ . With this assumption in place, the theorem follows once we establish that  $\pi_1(\Sigma(L))$  admits a left-ordering iff  $L$  is the unknot.

Present  $L$  by a connected, alternating diagram; color it, distinguish a root  $r$ , and let  $W$  denote the resulting white graph. It follows that every edge gets the same sign  $\epsilon$ . Mirroring  $L$  if necessary (which leaves  $\pi_1$  unchanged), we may assume that  $\epsilon = 1$ . Now suppose that  $\Gamma \cong \pi_1(\Sigma(L))$  possessed a left-ordering  $<$ . Choose a vertex  $v$  for which  $x_w \leq x_v$  for all  $w \in V$ . If  $x_v = x_w$  for all  $w \in V$ , then from the relation  $x_r$  it follows that  $1 = \Gamma \cong \pi_1(\Sigma(L))$ ; but then  $1 = |H_1(\Sigma(L))| = \det(L)$ , and since  $L$  is alternating, it follows that  $L = U$ .

Thus, we assume henceforth that  $L \neq U$  and seek a contradiction. It follows that there exists some  $w \in V$  for which  $x_w < x_v$ ; from the connectivity of  $W$ , we may assume that  $(v, w) \in E$ . It follows that  $1 < x_w^{-1}x_v$ , while  $1 \leq x_{w_i}^{-1}x_v$  for every other edge  $(v, w_i) \in E$ . Therefore, the product of all these terms in any order is greater than 1. In particular,  $1 < \prod_{i=1}^k (x_{w_i}^{-1}x_v) = r_v = 1$ , a contradiction.

□

### 3. DISCUSSION.

It remains an outstanding problem to relate  $\pi_1(Y)$  to the Heegaard Floer homology of a 3-manifold  $Y$ . As of this writing, it remains a possibility that a rational homology sphere  $Y$  is an L-space iff  $\pi_1(Y) \neq 1$  does not admit a left-ordering. Theorem 2.1 supports this conjecture, since  $\Sigma(L)$  is a rational homology sphere L-space for a non-split alternating link  $L$  (no confusion about the  $L$ 's!) [7, Prop.3.3]. Additional examples appear in [1, 2, 3, 4, 8].

In this spirit, Peter Ozsváth raises an interesting question. Let  $(Y_0, Y_1, Y_2)$  denote a surgery triple of rational homology spheres. That is, there exists a manifold  $M$  with torus boundary and a triple of slopes  $(\gamma_0, \gamma_1, \gamma_2)$  in  $\partial M$  such that  $Y_i$  results from filling  $M$  along slope  $\gamma_i$  and  $\gamma_i \cdot \gamma_{i+1} = +1$ , for all  $i \pmod{3}$ . Cyclically permuting the indices if necessary, assume that  $|H_1(Y_0)| = |H_1(Y_1)| + |H_1(Y_2)|$ .

**Question 3.1.** *If  $\pi_1(Y_0)$  admits a left-ordering, does it follow that one of  $\pi_1(Y_1)$  and  $\pi_1(Y_2)$  must as well?*

Note that if  $Y_1$  and  $Y_2$  are L-spaces, then so is  $Y_0$  according to the surgery exact triangle in  $\widehat{HF}$ . This is the motivation behind Question 3.1. An affirmative answer would imply that Theorem 2.1 extends to quasi-alternating links.

*Note.* Tetsuya Ito has applied the idea in this paper to a different presentation for  $\pi_1(\Sigma(L))$  to recover yet another proof of Theorem 2.1 [6].

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